

# Fine-grained Parameterized Algorithms

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## Problem set 1: Kernelization

**Overview:** In this problem set, you will develop and analyze kernelization algorithms.

**Instructions:** For each skill, select **exactly one** problem below and submit your solution in [Moodle](#); in your submission, make sure to repeat the problem that you are solving. The problems are roughly ordered by difficulty, choose a problem that you find non-trivial and interesting. (You are encouraged to try the other problems as well and ask us for feedback.)

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**!! Skill-1a. Falsify kernelization:** *I can precisely explain why a proposed kernelization algorithm for a given problem is not correct.*

**1.1 VC reduction rules for CVC.** The problem Connected Vertex-Cover is defined as follows: Given an undirected graph  $G$  and a positive integer  $k$ , decide whether there exists a vertex set  $C \subseteq V(G)$  such that  $C$  has size at most  $k$  vertices, the subgraph  $G[C]$  induced by  $C$  is connected, and  $C$  is a vertex-cover of  $G$ . Where do the kernelization rules for Vertex-Cover (see Theorem 2.4) fail when applied to Connected Vertex-Cover?

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**!! Skill-1b. Do-almost-nothing kernelization:** *I can design and prove the correctness of kernelization algorithms by arguing that every yes- or every no-instance already satisfies the size constraint.* (For an example, read Lemma 2.3 in [Cygan et al.](#) and how it is used, and the paragraph on p. 19 that talks about constant-size trivial instances.)

**1.2 Feedback vertex set in regular graphs.** Given an undirected graph  $G = (V, E)$ , a subset of vertices  $X \subseteq V$  is a *feedback vertex set* if removing  $X$  and all edges incident to  $X$  from  $G$  yields a forest. Show that the problem of deciding whether a graph has a feedback vertex set of size at most  $k$  has a kernel with  $O(k)$  vertices on regular, undirected graphs. (Note: The degree is not assumed to be constant!)

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**!! Skill-1c. High-degree reduction rules:** *I can design and prove the correctness of kernelization algorithms by using a suitable „high-degree“ type reduction rule.* (For an example, see Section 2.2.1 in [Cygan et al.](#))

**1.3 Line cover in the plane.** Consider the following problem: Given a set of  $n$  points in the plane, decide whether there is a set of  $k$  lines such that every point lies on at least one line. Prove that this problem has a kernel of size  $O(k^2)$ .

**1.4 Sunflower lemma for Set Packing.** First, read Section 2.6 about the Sunflower Lemma in the book, including its application to the  $d$ -hitting set problem. Then, using similar ideas, solve the following exercise on the  $d$ -set packing problem: You are given an integer  $k$  and a family of subsets  $\mathcal{A}$  of a universe  $U$ , where each set in  $\mathcal{A}$  is of size at most  $d$ . Decide whether there exist  $k$  sets  $S_1, \dots, S_k \in \mathcal{A}$  that are pairwise disjoint. Use the Sunflower Lemma to obtain a kernel for this problem with  $f(d) \cdot k^d$  sets, for some computable  $f$ .

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**Note:** The following problems are meant to deepen your understanding of the course material. They are not mandatory, but we encourage you to try them out. You can present your solutions in the exercise session, hand in a written solution for the assignment that focusses on formal writing, or ask us for feedback. Problems marked with 🍷🍷🍷 are especially challenging—they require that you invest a lot of time, play around with different ideas, and have a bit of luck that you find one that works. Best enjoyed with your favorite beverage and a friend. If you solve them, you can be proud of yourself. If you don't, you can still be proud of yourself for trying.

🍷🍷🍷 **1.5 Kernelization for Connected Vertex-Cover.** Consider the problem from 1.1.

- Show that the problem admits a kernel with at most  $2^k + O(k^2)$  vertices.
- Show that if the input graph does not contain a 4-cycle as a subgraph, then the problem admits a kernel with at most  $O(k^2)$  vertices.
- Extend the argument from b) to show that, for every fixed  $d \geq 2$ , Connected Vertex-Cover restricted to graphs that do not contain the biclique  $K_{d,d}$  admits a kernel with  $O(k^d)$  vertices.

**1.6 Details of the LP-based kernelization.** Let  $G = (V, E)$  be a graph, and let  $x \in \mathbb{R}^V$  be an optimum solution to the linear programming formulation of the vertex-cover problem, LPVC( $G$ ). While  $x$  is not necessarily half-integral, you will prove now that it can be turned into an optimal solution  $y$  that is half-integral. To this end, define  $y \in \mathbb{R}^V$  as follows:

$$y_v = \begin{cases} 0 & \text{if } x_v < \frac{1}{2}, \\ \frac{1}{2} & \text{if } x_v = \frac{1}{2}, \\ 1 & \text{if } x_v > \frac{1}{2}. \end{cases}$$

Prove that  $y$  is also an optimal solution to LPVC( $G$ ).

Hint: For  $\delta > 0$ , define  $\tilde{x} \in \mathbb{R}^V$  with

$$\tilde{x}_v = \begin{cases} x_v - \delta & \text{if } x_v < \frac{1}{2}, \\ \frac{1}{2} & \text{if } x_v = \frac{1}{2}, \\ x_v + \delta & \text{if } x_v > \frac{1}{2}. \end{cases}$$

Show for small enough  $\delta$  that  $\tilde{x}$  is also an optimal feasible solution to LPVC( $G$ ). Which setting of  $\delta$  leads  $\tilde{x}$  to have fewer values that are not half-integral? Now argue inductively.