Fine-grained Parameterized Algorithms

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Problem set 5: Treewidth and Graph minors

Overview: With this problem set, you can train reasoning about algorithmic meta-theorems and graph minor theory.

Instructions: For each skill, select **exactly one** problem below and submit your solution in Moodle; in your submission, make sure to repeat the problem that you are solving. The problems are roughly ordered by difficulty, choose a problem that you find non-trivial and interesting. (You are of course welcome to try the other problems as well and ask us for feedback.)

‼ **Skill-5a. Algorithmic meta-theorems:** *I can apply algorithmic meta-theorems to design fixed-parameter tractable algorithms.*

5.1 Maximum Cut using Courcelle's Theorem. A *cut* of a graph G is a partition (A, B) of the vertices of G and the *size* of a cut (A, B) is the number of edges of G that have one endpoint in A and one endpoint in B. The problem MaxCut asks, given a graph G, to compute the size of the largest cut in G. Prove that MaxCut can be solved in time $f(\mathsf{tw}(G)) \cdot |V(G)|$ for some computable function f by invoking Courcelle's Theorem (Theorem 7.11).

5.2 Graph Property Verification using Courcelle's Theorem. Let Φ be a graph property expressible in monadic second-order logic. The problem Verify[Φ] asks, given a graph G, to decide whether G has property Φ. Invoke Courcelle's Theorem (Theorem 7.11) to prove that this problem can be solved in time $f(\mathsf{vc}(G)) \cdot n$ for some computable function f, where $\mathsf{vc}(G)$ is the size of the smallest vertex-cover of G.

5.3 Cycle Packing using Graph Minors. In the *Cycle Packing* problem, we are given an undirected graph G and a positive integer k, and the goal is to check whether there exist k cycles in G that are pairwise vertex disjoint. Use Theorem 6.12ff from the book to prove that this problem is nonuniformly fixed-parameter tractable when parameterized by k.

5.4 Closest String using Integer Linear Programming. In the *Closest String* problem, we are given a set of k strings $x_1, \ldots, x_k$ over alphabet Σ, each of length L, and a positive integer d. The goal is to find a string y of length L such that the *Hamming Distance*[1] between y and x_i is bounded by d for every $i \in \{1, \ldots, k\}$. Use Theorem 6.5 from the book to prove that the problem is fixed-parameter tractable when parameterized by k and $|\Sigma|$.

Optional bonus task: Show that this problem parameterized by k only remains fixed-parameter tractable.

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[1] The Hamming Distance between two strings x and y of the same length is the number of positions i such that $x_i \neq y_i$ holds.

‼ **Skill-5b. Graph minors:** *I can mathematically reason about graph minors and apply the Graph Minors Theorem.* (See Section 6.3 and 7.7 in Cygan et al.)

5.5 Excluded Grid Theorem is Tight. Show that the dependency on k in the Excluded Grid Theorem needs to be $\Omega(k^2)$. That is, construct a graph of treewidth $\Omega(k^2)$ that does not contain a $k \times k$ grid as a minor.

5.6 Bidimensionality. Show that the following problems are bidimensional: Feedback Vertex Set, Induced Matching, Cycle Packing, Scattered Set for a fixed value of d, Longest Path, Dominating Set, and r-Center for a fixed r.

5.7 Brambles in Grid Graphs. Prove that for every $t > 1$, the $t \times t$ grid graph contains a bramble of order $t + 1$ and thus the treewidth of the $t \times t$ grid graph is t.